Some convergence results in discrete conformal geometry

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Workshop on Circle Packings and Geometric Rigidity

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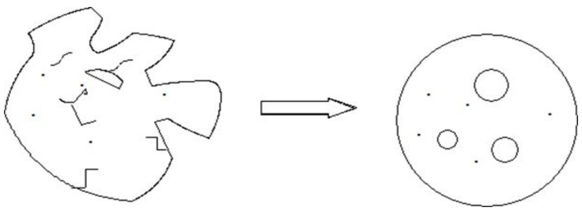
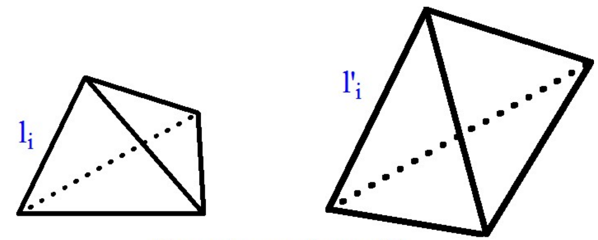
Outline

- Recall classical Riemann surfaces/conformal geometry
- Circle packing, Thurston’s convergence conjecture and rigidity
- Discrete conformal geometry from vertex scaling point of view
- Convergences in discrete conformal geometry
- Sketch of the proof
- Some problems on rigidity of infinite patterns

Riemann mapping theorem: every simply connected domain is conformal to **D** or **C**.





discrete conformal?

S = connected surface

Uniformization Thm(Poincare-Koebe) $\forall$ Riemannian metric d on S, $\exists\ \lambda: S \to \mathbf{R}_{>0}$ s.t., (S, λd) is a complete metric of curvature -1, 0, 1.

uniformization metric λd is conformal to d $\Longleftrightarrow$ angles in d and λd are the same

Q1. Can one compute the uniformization maps/metrics ?

Q2. Is there a discrete uniformization thm for polyhedral surfaces? ANS: yes (Gu-L-Sun-Wu)

Q3. Do discrete maps/metrics converge to the corresponding smooth counterparts?

Polyhedral surfaces

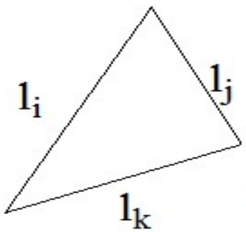
Triangulated PL surface

A PL metric d on (S,V) is a flat cone metric, cone points in V.

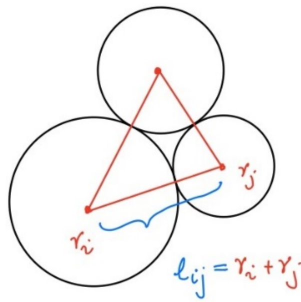
Isometric gluing of $\mathbf{E}^2$ triangles along edges: $(S, \mathcal{T}, l)$.

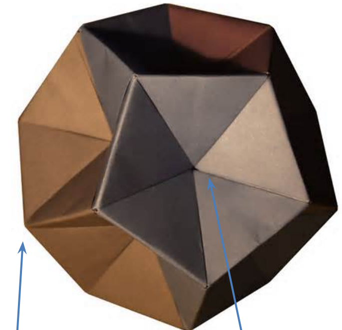
edge length

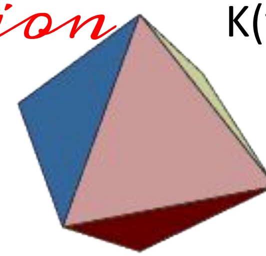
Triangulation



$l_i + l_j > l_k$







K(v)<0

K(v)>0

Eg. Circle packing metric r: V → $\mathbf{R}_{>0}$, $l_{ij} = r_i + r_j$

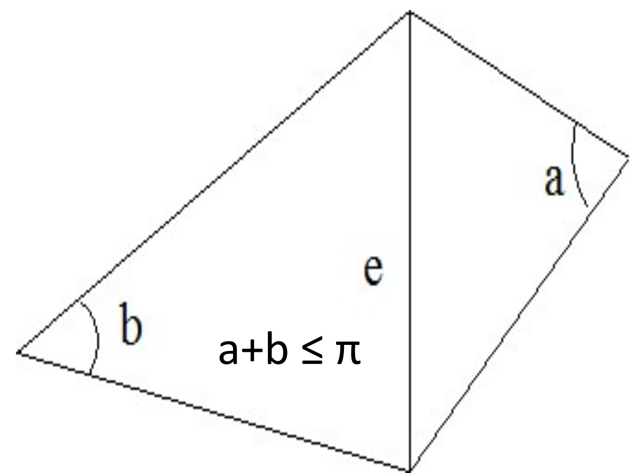
Curvature $K = K_d$: V → $\mathbf{R}$,

K(v)= 2π-sum of angles at v

= 2π- cone angle at v

A triangulated PL metric $(S, \mathcal{T}, l)$

is *Delaunay*: a+b ≤π at each edge e.

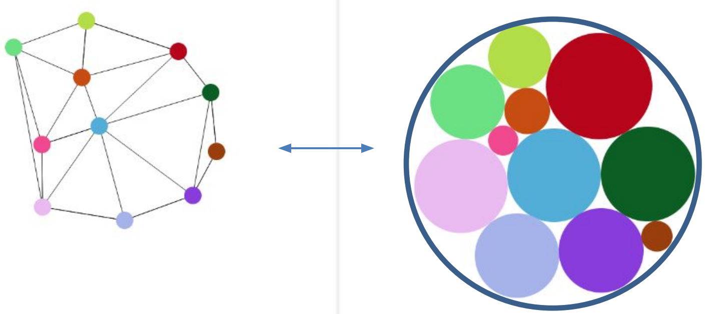


Discrete conformal geometry from circle packing point of view

Koebe-Andreev-Thurston theorem

Any triangulation of a disk is isomorphic to the nerve of a circle packing of the unit disk.

Discrete Riemann mapping



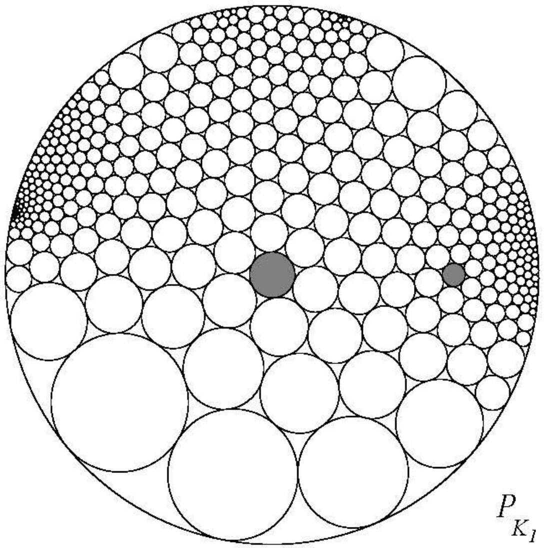
Thm (Thurston). For any simplicial triangulation $\mathscr{T}$ of a closed surface S of genus >1,
there ∃ ! a hyperbolic metric d and a circle packing P on (S, d) whose nerve is $\mathscr{T}$.

discrete uniformization theorem

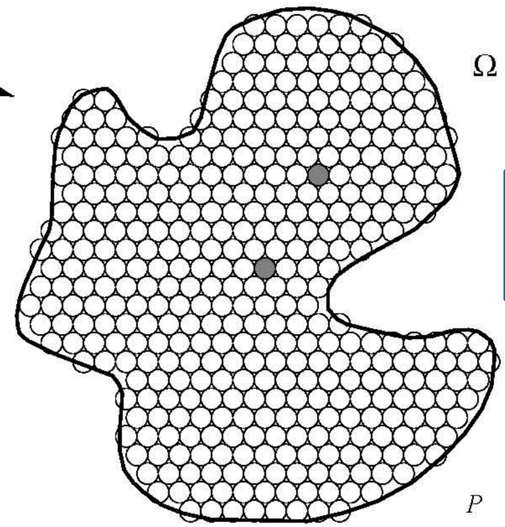
Circle packings produce a PL homeomorphism between the domains.

Question. Do they converge to the conformal map?

Thurston's discrete Riemann mapping conjecture,
Rodin-Sullivan's theorem

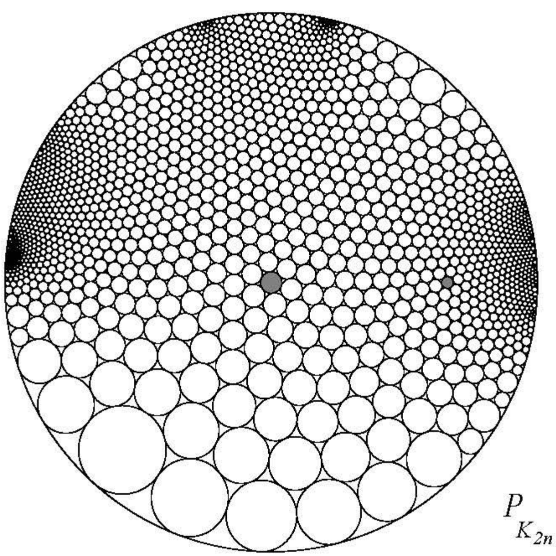


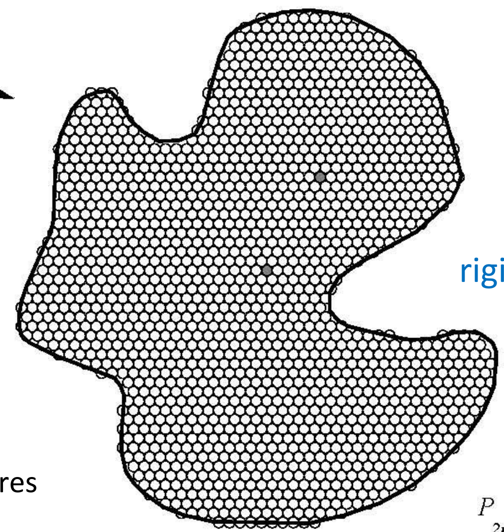
f_1 f_n



Koebe-Andreev-Thurston theorem

f_{2n}





Stephenson's pictures

$f_n \rightarrow$ Riemann mapping

Proof:

1. f_n converges

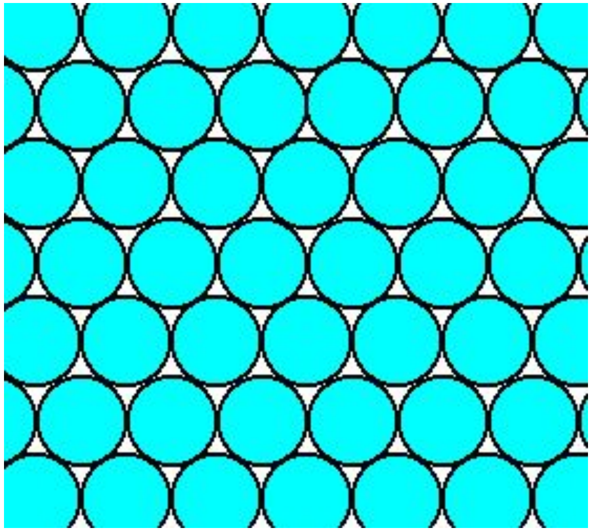
∃K, all f_n are K-quasi-conformal

2. limit is conformal

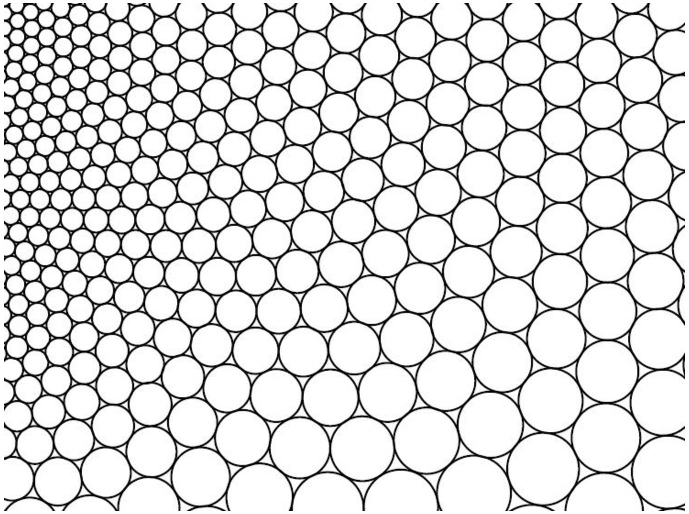
rigidity of the hexagonal circle packing

Rigidity of infinite circle packings

Hexagonal circle packing of **C**:



Regular



Convergence related to rigidity of infinite patterns

Thurston's Conjecture.

All hexagonal circle packings of **C** are regular.

Theorem (Rodin-Sullivan).

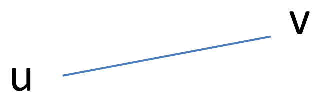
Thurston's conjecture holds.

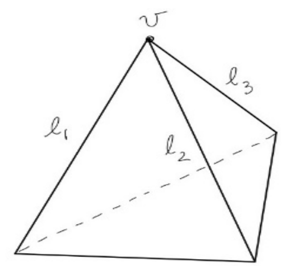
Thm (Schramm). If P and P' are two infinite circle packings of **C** whose nerves are isomorphic, then P and P' differ by a linear transformation.

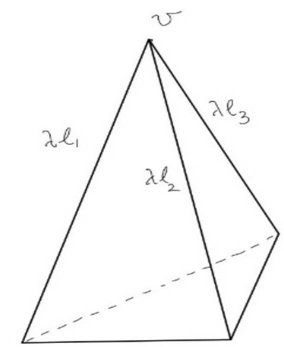
Discrete conformal geometry from vertex scaling point of view

Def. Two triangulated PL surfaces $(S, \mathcal{T}, l)$ and $(S, \mathcal{T}, \mathbf{l})$ are said to differ by a *vertex scaling* if $\exists\ \lambda: V(\mathcal{T}) \to R_{>0}$, s.t., $\mathbf{l} = \lambda_* l$ on E where

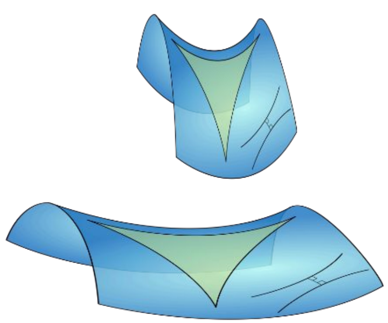
$$\lambda_* l(uv) = \lambda(u)\,\lambda(v)\, l(uv).$$







This is a discretization of the conformal Riemannian metric λg



$$g \leftrightarrow l$$

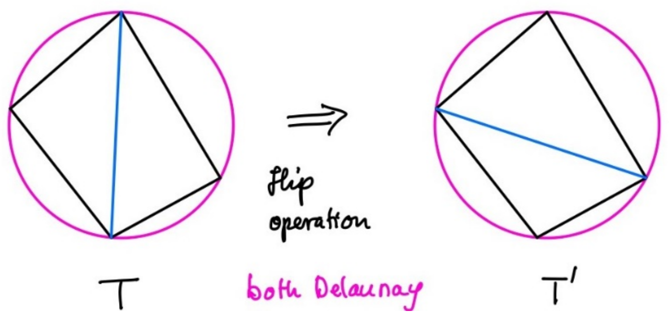
$$\lambda g \leftrightarrow \lambda_* l$$

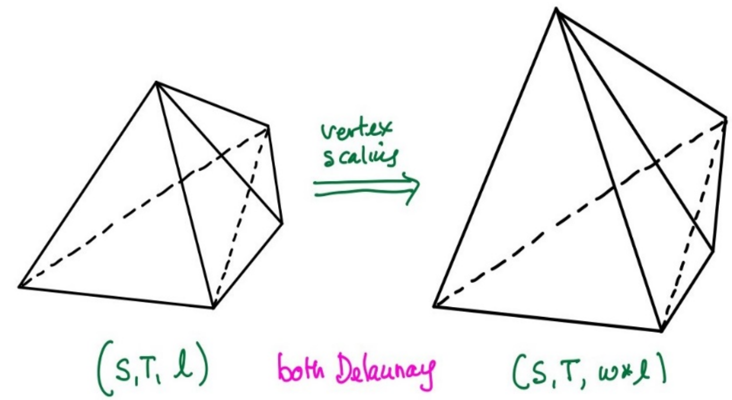
$$\left| d_{\lambda^4 g}(u, v) - \lambda(u)\, \lambda(v) d_g(u, v) \right| \le C d_g(u, v)^3$$ (Gu-L-Wu)

Discrete conformal equivalence of polyhedral metrics on (S,V)

Given a PL metric d on (S,V), find a *Delaunay triangulation T* of (S,V,d) s.t., d is (S, T, l).

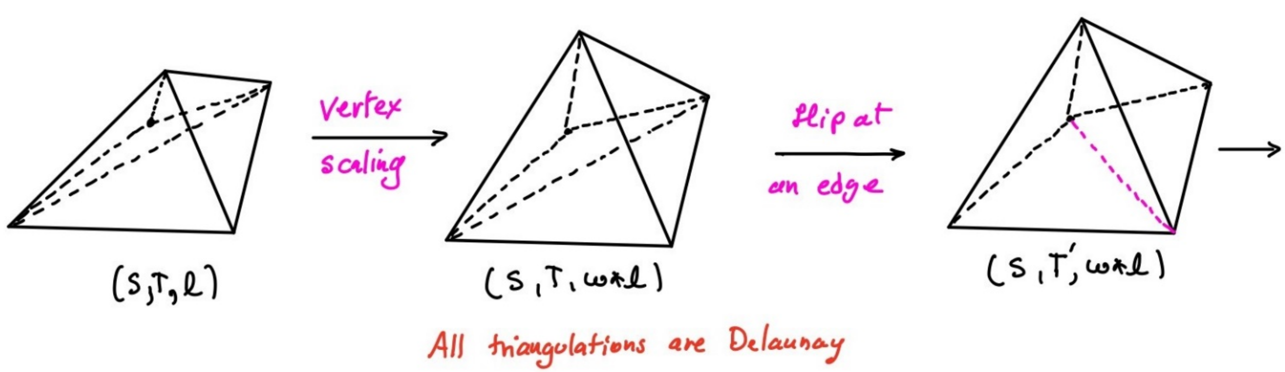
Move 1. Replace T by another *Delaunay* triangulation T' of (S,V,d).





Move 2. Replace (S, T, l) by a vertex scaled (S, T, w_*l) s.t. it is still *Delaunay*.

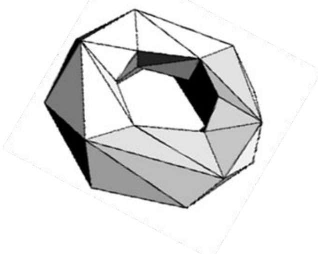
Def. (Gu-L-Sun-Wu) Two PL metrics d, d' on a closed marked surface (S,V) are *discrete conformal,* if they are related by a sequence of these two types of moves.

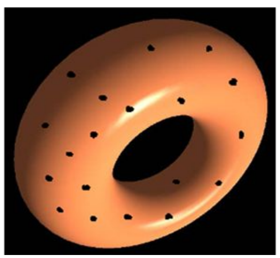


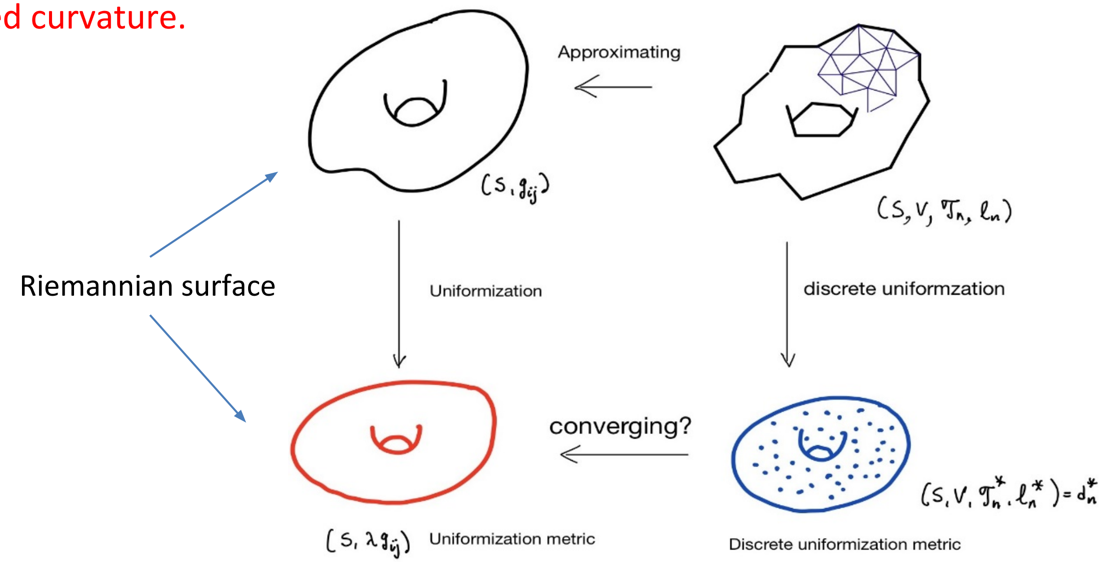
Thm (Gu-L-Sun-Wu). $\forall$ PL metric d on a closed (S,V) is discrete conformal to a unique (up to scaling) PL metric d* of constant curvature $\frac{2\pi\chi(S)}{|V|}$.

RM 1. First proved by Fillastre for the torus in a different content.

RM 2. It holds for any prescribed curvature.





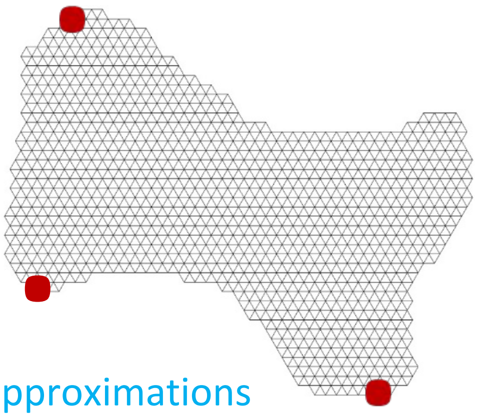


Question. Do the metrics d^*_n converge to the smooth uniformization metric?

Thm(Gu-L-Wu) . The convergence holds for any Riemannian torus ($S^1 X S^1$, g_{ij}).

Thm(Wu-Zhu 2020) . The convergence holds for any Riemannian closed surface of genus>1 in the hyperbolic background PL metrics.

Q. Do discrete conformal maps converge to the Riemann mapping?



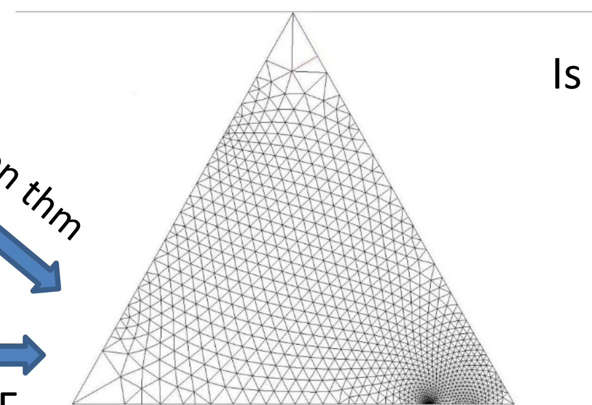
PL approximations

Convergence of $f_n \to$ Riemann mapping F?

quasi-conformality implies $f_n \to h$.

discrete uniformization thm

f_n



Is h conformal?

Riemann mapping F

work of
Bobenko-Pinkall-Springborn

Q. Is a Delaunay hexagonal triangulation of **C**, discrete conformal to the regular hexagonal triangulation, necessary regular?

Thm (L-Sun-Wu) . Given a Jordan domain Ω and $A,B,C \in \partial\Omega$, $\exists$ domains $\Omega_n \to \Omega$, s.t.,

(a) Ω_n triangulated by equilateral triangles,

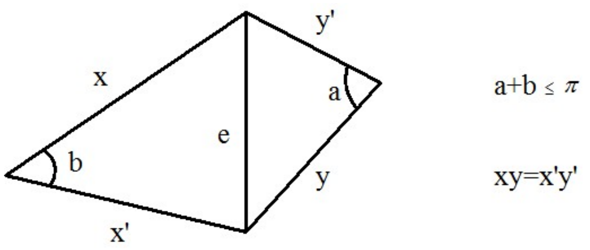
(b) the associated discrete uniformization maps $f_n \to$ Riemann mapping for $(\Omega;A,B,C)$.

Thm(L-Sun-Wu, Dai-Ge-Ma) . If T is a Delaunay geometric hexagonal triangulation of a simply connected domain in **C** s.t.,

$\exists$ g: V -> $\mathbf{R}_{>0}$ satisfying

length(vv')=g(v)g(v') for all edges e=vv',

then g = constant, i.e., T is regular.

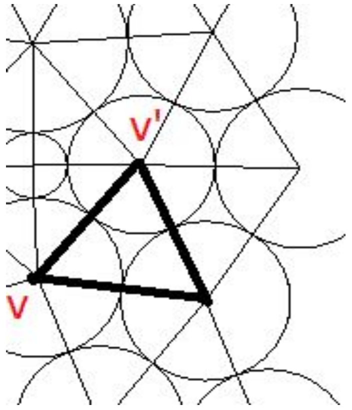


Thm (Rodin-Sullivan) . If T is a geometric hexagonal triangulation of a simply connected domain in **C** s.t.,

$\exists$ r: V -> $\mathbf{R}_{>0}$ satisfying

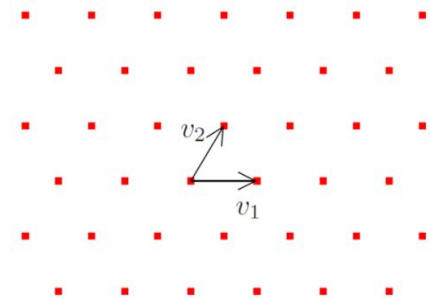
length(vv')=r(v)+r(v') for all edges e=vv',

then r=constant.

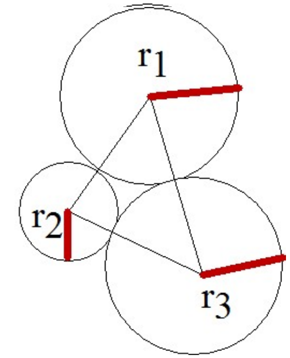


A new proof of Rodin-Sullivan's thm

Let V = **Z**+**Z**(η), $\eta = e^{\pi i/3}$:



$r_i = e^{u(v_i)}$



Thm(Rodin-Sullivan) If T is a geometric hexagonal triangulation of a simply connected domain in **C** s.t., ∃ u:V→ **R** satisfying

$$\text{length}(vv') = e^{u(v)} + e^{u(v')}, \text{ then } u = \text{const}.$$

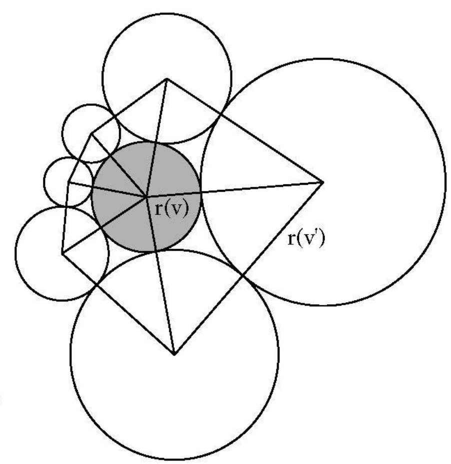
Liouville type thm. A bounded discrete harmonic function u on V is a constant.

Goal: for $\delta \in$ **V**, show $g(x)=u(x+\delta)-u(x)$ is constant.

Ratio Lemma (R-S). ∃ C >0 s.t., for all pairs of adjacent radii

$$r(v)/r(v') \le e^{C},$$

i.e., $|u(v)-u(v')| \le C$.



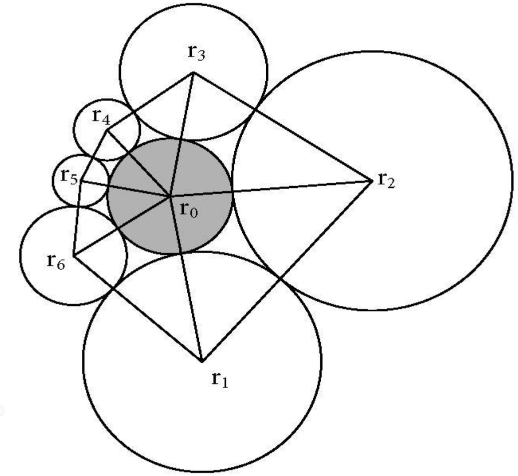
Corollary.

$$|u(v')| \le |u(v)| + Cd(v,v').$$

Max Principle: If $r_0 \ge R_0$ and $r_i \le R_i$, i=1,…,6, and

$K_r(v_0)=K_R(v_0)$,

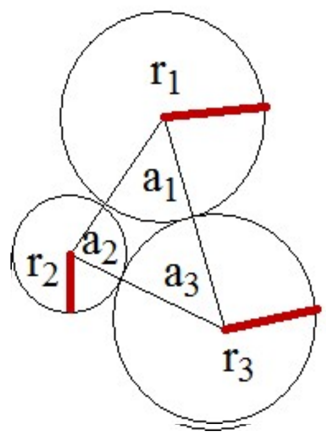
then $r_i=R_i$ for all i.

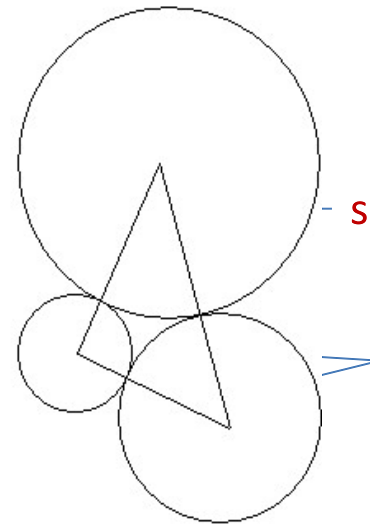


Proof (Thurston)

Fix r_2, r_3 and let $r_1 \nearrow$,

then $a_1 \searrow$ and $a_2 \nearrow$, $a_3 \nearrow$.





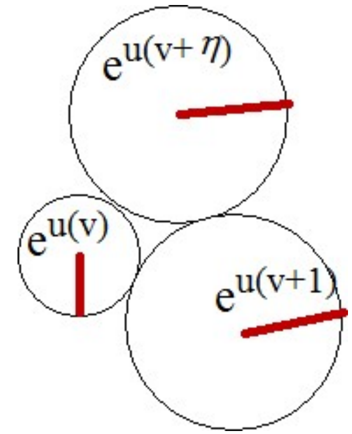
smaller

larger

Corollary. The ratio function r/R of two flat CP metrics has no max point unless r/R=constant.

A new proof of Rodin-Sullivan's thm, cont.

$V = \mathbf{Z} + e^{\pi i/3}\mathbf{Z}$.



Thm(Rodin-Sullivan). If T is a geometric hexagonal triangulation of a simply connected domain in $\mathbf{C}$ s.t., $\exists$ u:V -> $\mathbf{R}$ satisfying

$\text{length}(vv') = e^{u(v)} + e^{u(v')}$, then u=const.

Suppose $u: V \to \mathbf{R}$ is not a const.

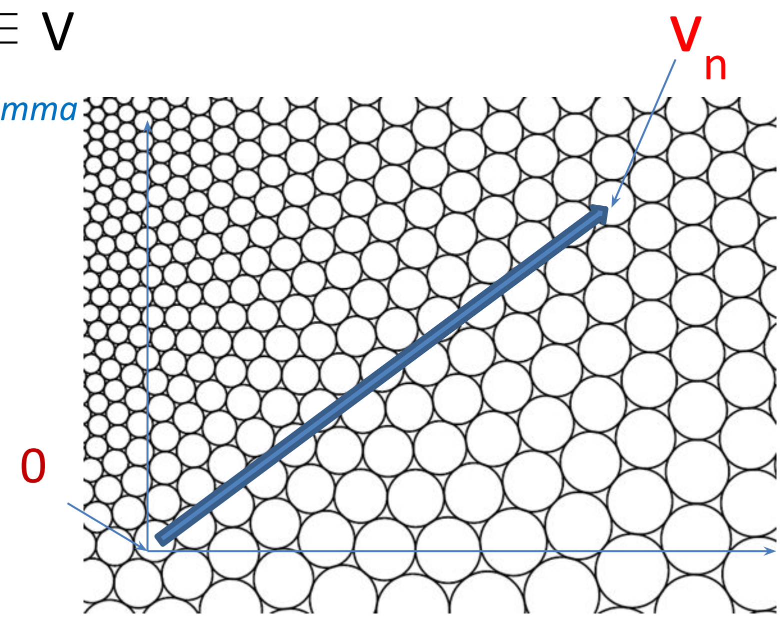
Then $\exists\, \delta \in \{1, e^{\pi i/3}\}$, s.t.,

$$\lambda = \sup\{ u(v+\delta) - u(v) : v \in V\} \neq 0 \text{ and } < \infty.$$

Take $v_n \in V$, s.t., $u(v_n+\delta) - u(v_n) > \lambda - 1/n$

$u(v+\delta) - u(v) \leq \lambda$, for all $v \in V$

$|u(v) - u(v')| \leq C$, $v \sim v'$, *ratio lemma*



Define, $u_n(v) = u(v+v_n) - u(v_n)$:

$u_n(0) = 0$,

$u_n(\delta) - u_n(0) > \lambda - 1/n$,

$u_n(v+\delta) - u_n(v) \leq \lambda$,

$|u_n(v)| \leq C\, d(v,0)$.

Combinatorial distance from v to 0.

Recall $\quad u_n(v)= u(v+v_n)-u(v_n) \in \mathbf{R}^V$:

$u_n(0)=0$, $\quad u_n(\delta)-u_n(0) >\lambda-1/n$, $\quad u_n(v+\delta)-u_n(v)\leq \lambda$, $\quad |u_n(v)| \leq C\, d(v,0)$.

Taking a subsequence, $\quad \lim_n u_n = u_\#$, $\; u_\# \in \mathbf{R}^V$, s.t.,

(1) the CP metric $e^{u\#}$ is still flat (may be incomplete).

(2) $\Delta u_\#(v) = u_\#(v+\delta)-u_\#(v)$ achieves maximum point at $v=0$.

By the *max principle*, $\quad u_\#(v+\delta)-u_\#(v) \equiv \lambda$.

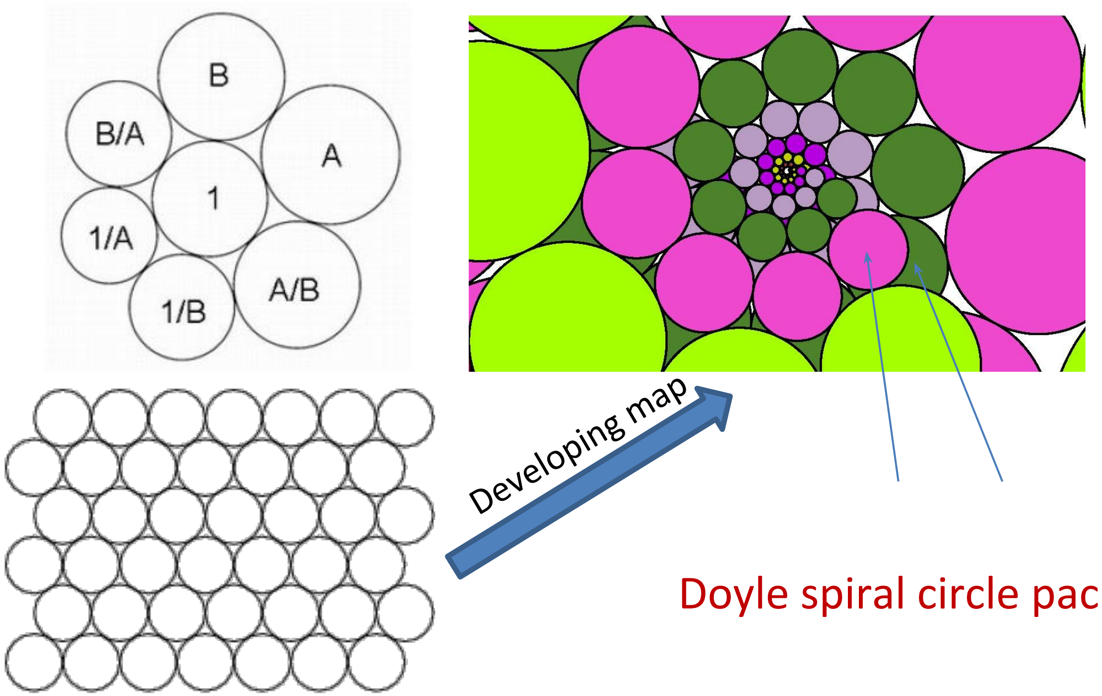
Repeat it for $u_\#$ (instead of u), taking limit to get $u_{\#\#}$. (δ, δ' generate V)

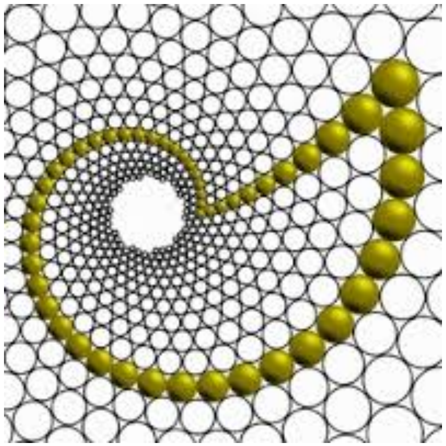
$$u_{\#\#}(v+\delta')-u_{\#\#}(v)= \text{constant}$$

$$u_{\#\#}(v+\delta)-u_{\#\#}(v) \equiv \lambda.$$

So $u_{\#\#}$ is a non-constant linear function on V.

F: V → R is linear if it is a restriction of a linear map on $\mathbf{R}^2$.





Doyle spiral circle packing (raduii=e^u, u linear, implies flat)

Lemma (Doyle) If f: V → R non-constant linear, then the CP metric e^f is *flat* and the developing map sends to two disjoint circles to two circles in **C** with overlapping interiors.

$\Rightarrow$ CP metric $e^{u\#\#}$ does not have injective developing map.

$\Rightarrow$ CP metrics $e^{u\#}$ and hence e^u do not have injective developing maps, a contradiction.

Need:

a ratio lemma (for taking limit),

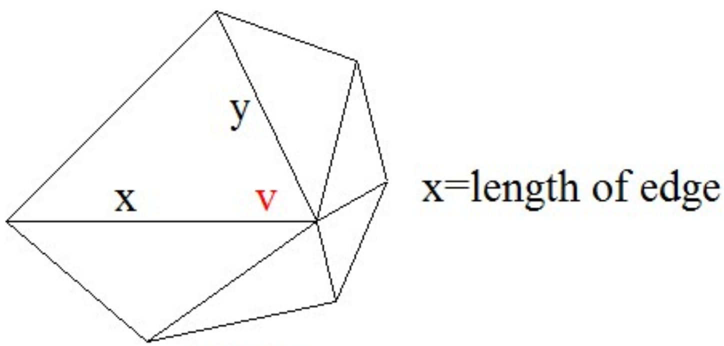
a maximum principle,

a spiral situation (log(radius) linear) producing self intersections.

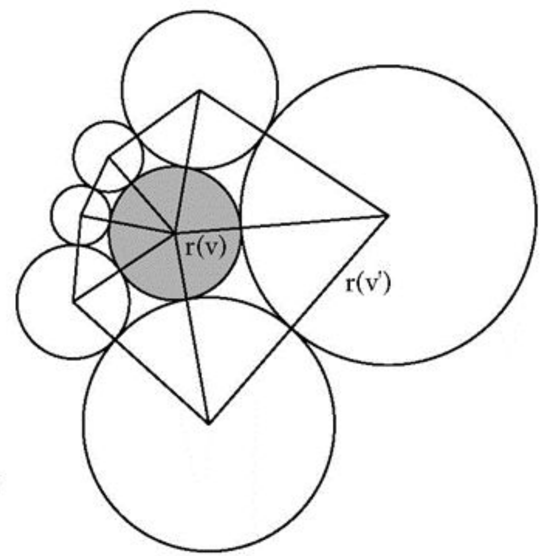
All of them hold in the vertex scaling setting.

L_0 is the constant function on the lattice V = $\mathbf{Z} + e^{\pi i/3}\mathbf{Z}$.

Ratio Lemma. If $w_* L_0$ is a PL metric s.t. K(v)=0, then $x/y \leq 6$.

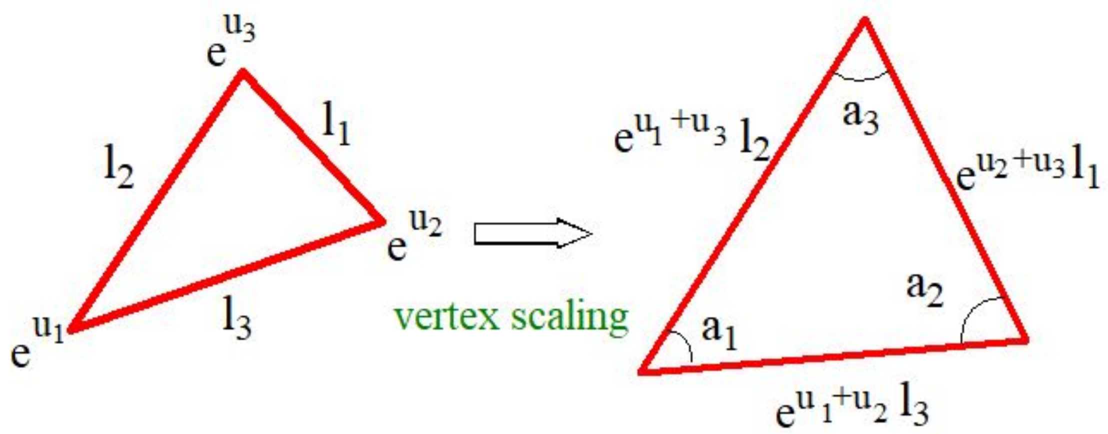


$B_1(v)$



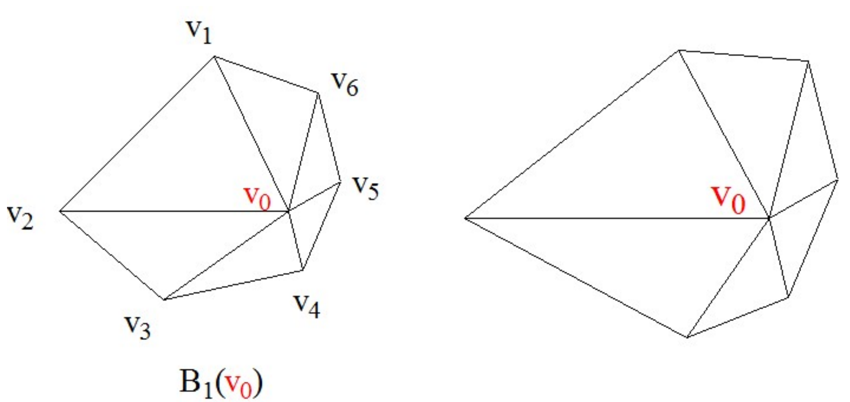
A maximum principle from a variational framework

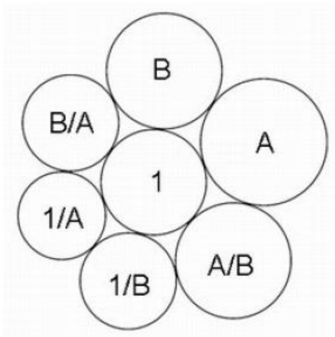
Prop (L, 2004).

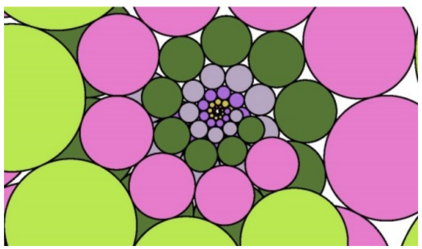


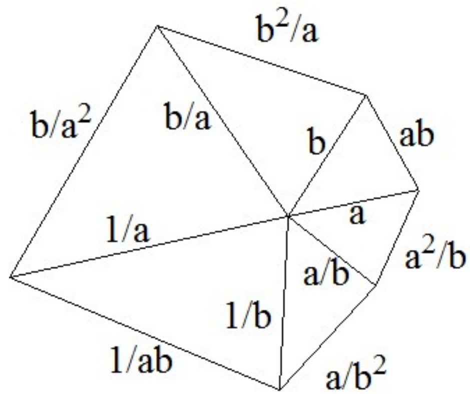
Then $\frac{\partial a_i}{\partial u_j} = \frac{\partial a_j}{\partial u_i}$ and $[\frac{\partial a_i}{\partial u_j}]_{3x3}$ is negative semi-definite.

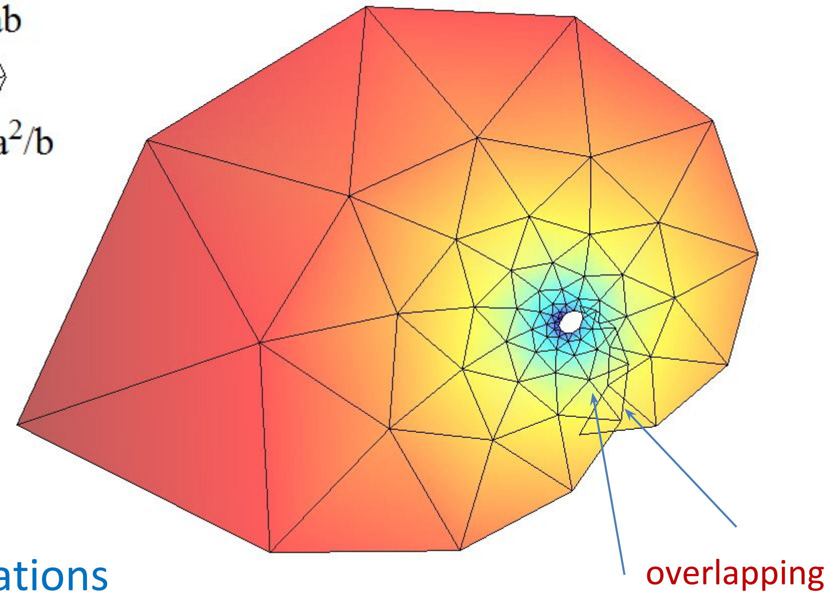
Maximum principle. Let $(B_1(v_0), l)$ and $(B_1(v_0), l')$ be two flat Delaunay PL metrics, s.t., $l' = u_* l$ and $u(v_0) = \max\{u(v_1), \ldots, u(v_6)\}$. Then u=constant.











Spiral triangulations

L_0 is a constant function on V.

Spiral Lemma (Gu-Sun-Wu). Suppose w: V → **R** is non-constant linear s.t. w_*L_0 is a piecewise linear metric on T. Then

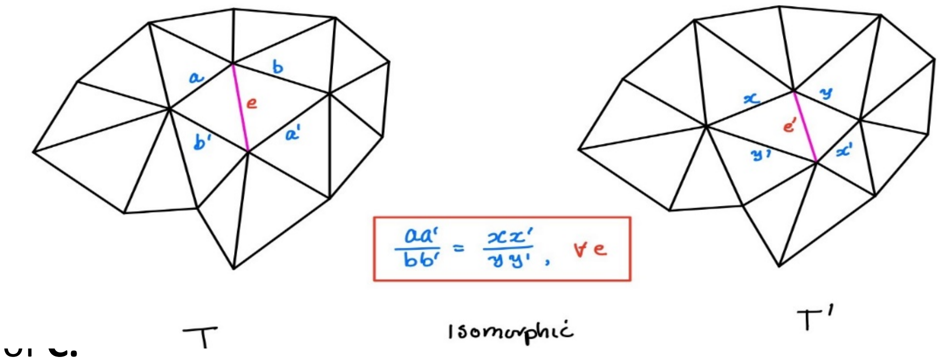
(1) w_*L_0 is flat, and

(2) ∃ two triangles in T whose images under the develop map intersect in their interiors.

Some conjectures on rigidity of infinite patterns

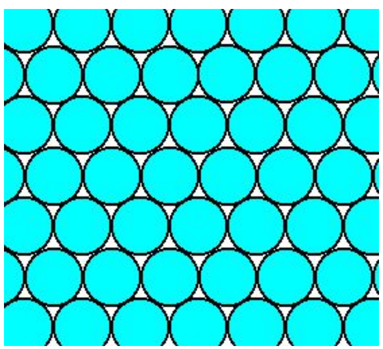
Conjecture (L-Sun-Wu). Suppose (**C**, V, T, l) and (**C**, V, T’, l') are two geometric triangulations of the plane s.t.,

1. both are Delaunay,
2. T, T’ are isomorphic topologically,
3. $w_* l = l'$.

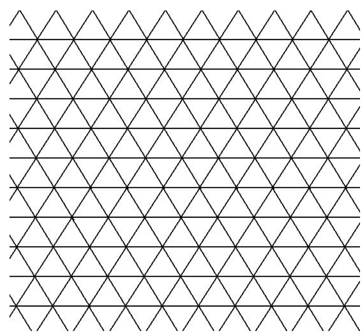


Then T and T’ differ by a linear transformation of **C**.

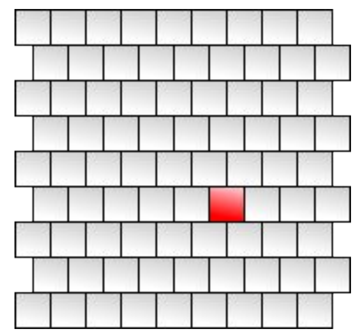
Counterpart of Schramm’s rigidity theorem.



Regular circle packing



Regular triangulation



Regular square tiling

Conjecture: If H is hexagonal square tiling of **C,** then all squares have the same size.

Thank you.

Conditions on triangulations to insure convergence

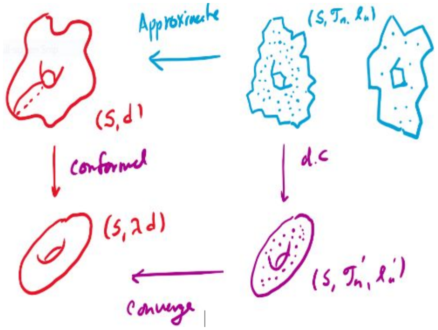
- Let (S,d) be a closed Riemannian surface of genus g, λd is the uniformization metric.

Goal: compute λd.

A sequence of triangulations (S, T_n, l_n) is *regular* if there exist $\delta>0$, $C>0$, $q_n \to 0$ s.t.

(1) all angles in T_n are in $(\delta, \frac{\pi}{2} - \delta)$,

(2) all lengths of edges in T_n are in $(\frac{q_n}{C}, Cq_n)$.



Thm (Gu-L-Wu). If (S, T_n, l_n) is a regular sequence of triangulated polyhedral tori approximating a Riemannian torus (S,d) and (S, T_n', l_n') is the flat polyhedral torus discrete conformal to (S, T_n, l_n) of area 1, then (S, T_n', l_n') converges uniformly to the uniformization metric λd associated to (S, d).

Wu-Zhu improved conditions:

Exists $\delta>0$ s.t., (1) all angles all in T_n are at least δ and

(2) sum of two angles facing each edge are at most $\pi - \delta$.